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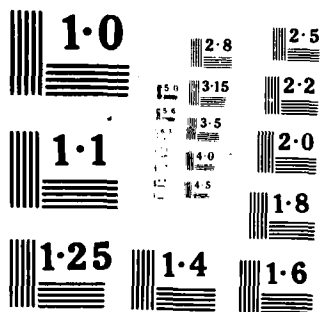
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## Differential Entropy and the Statistics of Instantaneous Failure

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# DIFFERENTIAL ENTROPY AND THE STATISTICS OF INSTANTANEOUS FAILURE

## I. Introduction

Instantaneous failure statistics are usually modelled in terms of a probability density  $f_T(t)$  of the (continuous) time to failure,  $0 \leq t < \infty$ , for a component or system under consideration [1,2]. In order to define a dimensionally consistent differential entropy for such a process, it is necessary to deal with a dimensionless probability density that is a function of a dimensionless time or ordering parameter. This may be accomplished by introducing a positive time scale parameter  $\tau_T$  so that the dimensionless time is just

$$\hat{t} \equiv t/\tau_T, \quad 0 \leq \hat{t} < \infty \quad (1.1)$$

The corresponding dimensionless probability density is

$$\hat{f}_T(\hat{t}) = \tau_T f_T(t) \quad (1.2)$$

Then it is straightforward to define a dimensionless differential entropy for failure statistics to be

$$\hat{S}_T \equiv - \int_0^\infty \hat{f}_T(\hat{t}) \log \hat{f}_T(\hat{t}) d\hat{t} = - E[\log \hat{f}_T(\hat{t})] \quad (1.3)$$

By explicit demonstration [3],  $\hat{S}_T$  is well-defined for all the customary models of instantaneous failure although  $\hat{S}_T$  is not necessarily positive.

One object of this paper is to point out a relationship between  $\hat{S}_T$  and the dimensionless relative failure rate at time  $\hat{t}$ , i.e. the dimensionless hazard rate,  $\hat{\lambda}_T(\hat{t})$ .

$$\hat{\lambda}_T(\hat{t}) \equiv \hat{f}_T(\hat{t})/[1-F_T(\hat{t})] \quad (1.4)$$

Here  $F_T(\hat{t})$  is the probability distribution at time  $\hat{t}$ . It is clear that  $\hat{\lambda}_T(\hat{t})d\hat{t}$  is just the ratio of the probability of failure in  $[\hat{t}, \hat{t}+d\hat{t}]$  to the probability of survival to time  $\hat{t}$ . It is established in the next section that  $\hat{S}_T$  is equal to the negative of the expectation of the logarithm of the hazard function, i.e. to  $-E[\log \hat{\lambda}_T(\hat{t})]$ , plus unity.

This relationship is then used as a basis of a discussion of the behavior  $\hat{S}_T$  under transformations of ordering parameter, which may occur because of change of physical conditions. Transformations that leave  $\hat{S}_T$  invariant leave  $E[\log \hat{\lambda}_T(\hat{t})]$  invariant. Such invariance is interpreted as meaning that the conditions of failure remain unchanged even if measured in terms of a different ordering parameter. Similarly, the lack of invariance of  $\hat{S}_T$  or  $E[\log \hat{\lambda}_T(\hat{t})]$  is interpreted to mean the conditions of failure have been changed as e.g. in accelerated testing. Both types of behavior are discussed further in section III.

## II. Differential Entropy and the Expectation of the Logarithm of the Hazard Rate

As indicated in the introduction, there is a simple relationship between the differential entropy  $\hat{S}_T$  and the expectation of the logarithm of the hazard rate  $E[\log \hat{\lambda}_T(\hat{t})]$ . That relationship is based on the following observations. First note that the cumulative hazard rate  $\hat{\Lambda}_T(\hat{t})$  is just

$$\hat{\Lambda}_T(\hat{t}) = \int_0^{\hat{t}} \hat{\lambda}_T(\hat{t}') d\hat{t}' = -\log[1-F_T(\hat{t})] \quad (2.1)$$

Further note that the cumulative probability weighted with the cumulative hazard rate is independent of the choice of probability density.

$$\begin{aligned} \Omega(F_T) &\equiv - \int_0^{\hat{t}} \hat{f}_T(\hat{t}') \log[1-F_T(\hat{t}')] d\hat{t}' \\ &= - \int_0^{F_T} \log(1-F'_T) dF'_T \end{aligned}$$

$$= (1-F_T)\log(1-F_T) + F_T \quad (2.2)$$

In particular when  $F_T(\hat{t}) = 1$ , i.e. as  $\hat{t} \rightarrow \infty$ ,

$$\Omega(1) = 1 \quad (2.3)$$

independent of the choice of  $\hat{f}_T(\hat{t})$ .

It follows immediately from the definitions of  $\hat{S}_T$  and  $\hat{\lambda}_T(\hat{t})$  respectively in (1.3) and (1.4) that

$$\hat{S}_T = -E\{\log \hat{\lambda}_T(\hat{t})\} + 1 \quad (2.4)$$

Hence an evaluation of  $\hat{S}_T$  for a particular probability density serves equally well as an evaluation of  $E\{\log \hat{\lambda}_T(\hat{t})\}$ . Thus the differential entropy takes on a very specific empirical meaning in the context of the statistics of instantaneous failures. This empirical basis is used in the next section to interpret the invariance properties (or lack thereof) for  $\hat{S}_T$  under transformations of ordering parameter.

### III. Invariance Properties of the Differential Entropy

In general a time or ordering parameter is a positive cumulative function that increases monotonically from an origin. A transformation of ordering parameter which may arise from changes in physical conditions is taken to mean a one-to-one relationship between two ordering parameters such that the derivative of the transformed parameter with respect to the original parameter is positive and finite almost everywhere, i.e., except possibly at isolated points. Implicit in the specification in section I of the range of the ordering parameter  $\hat{t}$  to be  $0 < \hat{t} < \infty$  is the setting of origin for failures at  $\hat{t} = 0$ . Transformed ordering parameters will here also be assumed to be aligned in the sense that they all share the same origin.

Consider then the transformation  $\hat{u} = \hat{u}(\hat{t})$ ,  $\hat{u}(0) = 0$ , where  $\hat{u}$  involves the ratio of a dimensional ordering parameter to a corresponding time scale



parameter  $\tau_U$ . In general, such transformations are not linear. The one-to-one nature of the transformation means that

$$F_U(\hat{u}) = F_T(\hat{t}) , \hat{u} = \hat{u}(\hat{t}) . \quad (3.1)$$

or

$$\hat{f}_U(\hat{u}) = \hat{f}_T(\hat{t}) / [d\hat{u}(\hat{t})/d\hat{t}] \quad (3.2)$$

Then

$$\hat{S}_U = \hat{S}_T + \int_0^\infty \hat{f}_T(\hat{t}) \log[d\hat{u}(\hat{t})/d\hat{t}] d\hat{t} \quad (3.3)$$

In general the integral on the right hand side of (3.3) does not vanish so that the differential entropy is not invariant under a transformation of ordering parameter.

In view of (2.4), (3.3) may be rewritten as

$$E[\log \hat{\lambda}_U(\hat{u})] = E[\log \hat{\lambda}_T(\hat{t})] - \int_0^\infty \hat{f}_T(\hat{t}) \log[d\hat{u}(\hat{t})/d\hat{t}] d\hat{t} \quad (3.4)$$

Thus  $E[\log \hat{\lambda}_T(\hat{t})]$  is also not in general invariant under the transformation determined by (3.1). Such a lack of invariance is not unreasonable and, in fact, should be expected when the conditions of failure are changed. Changes in the condition of failure are a usual occurrence, for example, when an object being observed for failure is subjected to reduced or increased ambient temperature in the course of a life test. Consider the simple case of a linear transformation

$$\hat{u}(\hat{t}) = \hat{a}\hat{t} \quad (3.5)$$

where  $\hat{a}$  is a positive dimensionless constant. The transformation can either be viewed as a change in the dimensional ordering parameter from  $t$  to  $\hat{a}t$  or a

change in the time scale parameter from  $\tau_T$  to  $\tau_U = \tau_T/\hat{a}$ . If  $\hat{a} = 1$ , there are compensating changes respectively in the dimensional ordering and scale parameters e.g.  $\hat{t} = kt/k\tau_T$  for arbitrary  $k$ , and there is no change in the differential entropy. If  $\hat{a} > 1$ ,  $\hat{u}(\hat{t})$  is larger than  $\hat{t}$  so that the process must be slowed to accommodate (3.1). Similarly if  $\hat{a} < 1$ , the failure process must be accelerated to accommodate (3.1). This variation of behavior with the magnitude of  $\hat{a}$  is also revealed by considering the behavior of  $E[\log \hat{\lambda}_T(\hat{t})]$  under the transformation (3.5). In that case

$$E[\log \hat{\lambda}_U(\hat{u})] = E[\log \hat{\lambda}_T(\hat{t})] - \log \hat{a} \quad (3.6)$$

Again it is clear that the failure process is slowed for  $\hat{a} > 1$  and accelerated for  $\hat{a} < 1$ .

To go beyond the simple situation of a linear transformation, it is instructive to consider specific probability distributions for the failures. Two of the most usual are the exponential and the Weibull with respective distributions,

$$F_T = 1 - \exp(-\lambda_T t) \quad (3.7a)$$

$$F_W = 1 - \exp(-(\eta_W w)^\beta), \quad \beta > 0 \quad (3.7b)$$

Here  $t$  and  $w$  are dimensional ordering parameters and  $\lambda_T$  and  $\eta_W$  are dimensional rate parameters. (For the exponential distribution  $\lambda_T$  is the dimensional hazard rate.) By introduction of  $\tau_T$  and  $\tau_W$  respectively, (3.7a,b) are readily expressed in terms of the dimensionless quantities  $\hat{\lambda}_T = \tau_T \lambda_T$ ,  $\hat{t} = t/\tau_T$ ,  $\hat{\eta}_W = \tau_W \eta_W$  and  $\hat{w} = w/\tau_W$ . Consider then the transformation

$$t = b w^\beta \text{ or } \hat{t} = \hat{b} \hat{w}^\beta \quad (3.8)$$

where  $b$  has the dimensions of  $[\text{time}]^{1-\beta}$ ,  $\hat{b}$  is dimensionless, and both are positive. From (3.1) and (3.7a,b), it follows that

$$b = \eta_W^\beta / \lambda_T, \quad \hat{b} = \hat{\eta}_W^\beta / \hat{\lambda}_T \quad (3.9)$$

The difference between the differential entropies calculated respectively for the exponential and Weibull distributions is just

$$\hat{S}_T - \hat{S}_W \equiv \Delta \hat{S}_{TW} = \log \beta \hat{\eta}_W / \hat{\lambda}_T - \gamma(\beta-1)/\beta \quad (3.10)$$

Here  $\gamma$  is the Euler constant,  $\gamma = 0.577215\dots$ . Clearly, depending on the values of  $\beta$  and the ratio  $\hat{\eta}_W / \hat{\lambda}_T$ ,  $\Delta \hat{S}_{TW}$  can be either positive or negative corresponding respectively to decelerated or accelerated failures. The latter behavior is again immediately revealed by the fact that  $\Delta \hat{S}_{TW}$  is the negative of the difference in the expectation of the difference in the logarithms of the hazard rate for the corresponding exponential and Weibull distributions.

When  $\Delta \hat{S}_{TW} = 0$ , the differential entropy remains invariant under the transformation from an exponential to a Weibull distribution. This additional condition serves to remove some of the arbitrariness in the choices of  $\tau_T$  and  $\tau_W$ . Indeed if a choice is made for one of the latter two, the other becomes (in general) a function of  $\eta_W$  and  $\lambda_T$ . Actually it is convenient to make the latter dependence tacit by choosing

$$\tau_T = \lambda_T^{-1} \text{ or } \hat{\lambda}_T = 1 \quad (3.11)$$

so that the invariant  $\hat{S}_T = \hat{S}_W = 1$ . Then (3.10) for  $\Delta \hat{S}_{TW}=0$  yields

$$\hat{\eta}_W = \beta^{-1} \exp(1-\beta^{-1})\gamma \equiv \hat{A}(\beta) \quad (3.12)$$

Here the right hand equality serves to define  $\hat{A}(\beta)$  so that

$$\tau_W = \hat{A}(\beta) \eta_W^{-1} \quad (3.13)$$

which determines the dimensional scale parameter in terms of the rate parameter  $\eta_W$ . It follows directly from (3.9) and (3.12) that

$$\hat{b} = [\hat{A}(\beta)]^\beta = b\tau_W^\beta/\tau_T \quad (3.14)$$

or

$$\tau_T = b[\tau_W/\hat{A}(\beta)]^\beta \quad (3.15)$$

Thus the respective dimensional time scales are related by the same transformation (3.8) as the dimensional order parameters provided there is an appropriate normalization determined by the condition  $\Delta\hat{S}_{TW}=0$ .

The condition  $\Delta\hat{S}_{TW}=0$  may be understood to mean that the physical mechanisms of failure are not changed. For example, one could take the view that an object has its own internal clock which orders its evolution to failure. The clock used in the laboratory for measurements in general will be different from the internal clock. Since in such a situation, the physical mechanisms of failure remain the same independent of the ordering parameter used for enumeration of failures,  $E[\log\hat{\lambda}_T(\hat{t})]$  and hence also  $\hat{S}_T$  would remain invariant. When this invariance obtains under the transformation (3.8) as for example when the measured distribution is Weibull with an underlying exponential, the measured failure rate  $\eta_W$  should have some surprising features. For instance if  $\beta>1$ , it should be smaller than expected and similarly if  $\beta<1$ , it should be larger than expected. Another possible indicator of the presence of an underlying distribution here would be unexpected dependencies of the rate on system specific parameters. For this example, if the change of a power of parameter to a (power)/ $\beta$  provided expected behavior, then one would have an indicator that there is an underlying exponential distribution. This latter indicator also provides the key to improved performance by its revelation respectively of an increased or decreased (depending on the value of  $\beta$ ) sensitivity of parameters. Appropriate modification of the parameters entering in the rate parameter itself, could lead to improved performance.

The example given here exhibited a relationship between the exponential distribution into the Weibull distribution by means of a transformation, Eq. (3.8). This discussion suggests a useful general method of finding relationships between different distribution functions and their respective entropies. Two other examples of interest in Statistics of Instantaneous Failures would be (a)  $\beta$ -Weibull to  $\beta'$ -Weibull,  $\beta \neq \beta'$  and (b) exponential to bathtub distribution.

The case of transformation of a Weibull to another Weibull is of some interest. In the notation of (3.10) it is found that

$$\hat{S}_W - \hat{S}_{W'} \equiv \Delta \hat{S}_{WW'} = \log(\beta' \hat{\eta}_W / \beta \hat{\eta}_W) + (\beta - \beta' / \beta \beta') \gamma.$$

The discussion of the type following (3.6) in this case shows that the failure process is slowed or accelerated depending on the relative values of the rates  $\hat{\eta}_W$  and  $\hat{\eta}_{W'}$  as well as the exponents  $\beta$  and  $\beta'$ .

The discussion above concerned transformation which carried the range of the ordering parameter variable from  $(0, \infty)$  to  $(0, \infty)$ . It is also possible to make other types of transformations which carry  $(0, \infty)$  to  $(a, b)$  or to  $(-\infty, \infty)$ . In a generalized sense, these may also be considered as transformations of time but the physical meaning is perhaps obscured by such transformations. Thus, we could use the transformation  $\hat{\lambda} \hat{x} = \log(1 + e^{\hat{b} \hat{y}})$  ( $\hat{b} > 0$ ) which carries  $\hat{x}:(0, \infty)$  to  $\hat{y}:(-\infty, \infty)$ . and the exponential distribution in  $\hat{x}$  is then transformed to a logistic distribution  $e^{-\hat{b} \hat{y}} / (1 + e^{-\hat{b} \hat{y}})^2$ . The entropy change  $\hat{S}_X - \hat{S}_Y$  is  $(-1 - \ln(\hat{\lambda} / \hat{b}))$ . The transformation  $\hat{\lambda} \hat{x} = \log(\hat{b} - \hat{a} / \hat{b} - \hat{z})$ ,  $\hat{a} < \hat{z} < \hat{b}$ ,  $\hat{x}:(0, \infty)$  becomes  $\hat{z}:(\hat{a}, \hat{b})$  and the exponential distribution becomes a uniform distribution in  $\hat{z}$ . The change in entropy,  $\hat{S}_X - \hat{S}_Z$  is  $1 - \ln[\hat{\lambda}(\hat{b} - \hat{a})]$ .

The bathtub distribution is obtained from an exponential distribution when the transformation

$$\hat{\lambda} \hat{t} = e^{(\hat{\gamma} \hat{z})^b} - 1, \quad \hat{\gamma}, b > 0$$

is made on the exponential distribution in the  $\hat{t}$ -variable. This takes  $\hat{t}:(0,\infty)$  to  $\hat{z}:(0,\infty)$  and the exponential distribution is transformed to the bathtub distribution in the  $\hat{z}$ -variable. The entropy difference is given by

$$\hat{S}_Z - \hat{S}_T = -\ln\left(\frac{\hat{\gamma}b}{\lambda}\right) - \int_0^\infty e^{-y} \ln(1+y) dy - \frac{(b-1)}{b} \int_0^\infty e^{-y} dy \ln(\ln(1+y))$$

In summary, we have related the dimensionless entropy for failure statistics to the negative of the expectation of the logarithm of the hazard rate. We have also pointed out that the general lack of invariance of the differential entropy under a transformation may be viewed as equivalent to changing the physical conditions under which the hazard function is measured. In future work, we plan to use this general observation as a basis for the development of strategies for accelerated testing.

### References

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